

Bosonization of Supersymmetric KdV equation

Xiao Nan Gao¹ and S Y Lou^{1,2,3}

¹*Department of Physics, Shanghai Jiao Tong University, Shanghai, 200240, China*

²*Faculty of Science, Ningbo University, Ningbo, 315211, China*

³*School of Mathematics, Fudan University, Shanghai, 200433, China*

(Dated: August 23, 2011)

Abstract

Bosonization approach to the classical supersymmetric systems is presented. By introducing the multi-fermionic parameters in the expansions of the superfields, the $\mathcal{N} = 1$ supersymmetric KdV (sKdV) equations are transformed to a system of coupled bosonic equations. The method can be applied to any fermionic systems. By solving the coupled bosonic equations, some novel types of exact solutions can be explicitly obtained. Especially, the richness of the localized excitations of the supersymmetric integrable system are discovered. The rich multi-soliton solutions obtained here have not yet been obtained by using other methods. Unfortunately, the traditional known multi-soliton solutions can also not be obtained by the bosonization approach of this paper. Some open problems on the bosonization of the supersymmetric integrable models are proposed in the both classical and quantum levels.

A. Introduction

It is known that the supersymmetric integrable systems are very important in many physical fields especially in quantum field theory and cosmology such as superstring theory where it appears as a basic part of the string worldsheet physics or the theory of two-dimensional solvable lattice models, e.g., tricritical Ising models [1, 2]. Though the supersymmetric integrable systems have been studied by many authors in the both quantum and classical levels, various important problems are still open. For instance, in the usual quantum field theory, the bosonization approach is one of the powerful methods which simplifies the procedure to treat complex fermionic fields [3]. However, in our knowledge, there is no method to find a proper bosonization procedure for both quantum and classical supersymmetric integrable models. To treat the integrable systems with fermions such as the super integrable systems [4], supersymmetric integrable systems [5] and pure integrable fermionic systems [6] is much more complicated than to study the integrable pure bosonic systems. Therefore, it is significant if one can establish a proper bosonization procedure to treat the supersymmetric systems even if in the classical level.

In this paper, taking the $N = 1$ classical supersymmetric KdV (sKdV) system as a simple example we propose a simple bosonization approach to find exact solutions of supersymmetric systems. Actually, the method has been used by Andrea et al [7] to obtain new integrable bosonic systems. Here, we apply the method to find new exact solutions of supersymmetric integrable systems. One essential advantage of the method is that it can effectively avoid difficulties caused by intractable fermionic fields which are anticommuting. The $N = 1$ supersymmetric versions of the Korteweg-de Vries equation have been found more than 20 years ago [8–10], which are the beginning of the field of supersymmetric integrable systems. The far-reaching significance lies not only in mathematics, but also in the applications in various areas of modern theoretical physics. Therefore, investigating their properties and searching for their exact solutions are of great importance and interest.

For the integrable sKdV system in the sense of possessing a Lax pair, many remarkable properties have been discovered, such as the Painlevé property [11], the bi-Hamiltonian structures [12, 13], the Darboux transformation [14], the bilinear forms [15, 16], the Bäcklund transformation (BT) [18], the Lax representation [19] and the nonlocal conservation laws [20]. Some types of multisoliton solutions are also known for the integrable sKdV system [15–19]. However, because anticommutative fermionic fields bring some difficulties in dealing with supersymmetric

equations, to get exact solutions of the supersymmetric systems is, especially, much more difficult than the usual pure bosonic systems.

The $\mathcal{N} = 1$ supersymmetric version of the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1)$$

is established by extending the classical spacetime (x, t) to a super-spacetime (θ, x, t) , where θ is a Grassmann variable, and the field u to a fermionic superfield

$$\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t), \quad (2)$$

which leads to a nontrivial result [9]

$$\Phi_t + 3(\mathcal{D}\Phi_x)\Phi + 3(\mathcal{D}\Phi)\Phi_x + \Phi_{xxx} = 0, \quad (3)$$

where $\mathcal{D} = \partial_\theta + \theta\partial_x$ is the covariant derivative. The component version of Eq. (3) reads

$$u_t + u_{xxx} - 3\xi\xi_{xx} + 6uu_x = 0, \quad (4a)$$

$$\xi_t + \xi_{xxx} + 3u_x\xi + 3u\xi_x = 0, \quad (4b)$$

where u and ξ are bosonic and fermionic component fields, respectively. Vanishing ξ in Eq. (4), only the usual classical KdV equation remains.

Previous studies of the sKdV system were all directly based on Eq. (3) or (4). In this paper, we are only concentrated on bosonization of the sKdV equations by expanding the superfields with respect to the multi-fermionic parameters. In the next section, we present the bosonization approach of the sKdV system, in which the superfields are expanded about two fermionic parameters. And then the general traveling (in the usual space-time) periodic wave solutions, including the solitary waves as special cases, of the model are found. some special types of nontraveling (in the usual space-time) wave solutions (including all possible exact solutions of the usual KdV equation) are also obtained. In sections C and D, we extend the bosonization approach of the sKdV system to the case of three fermionic parameters and n fermionic parameters respectively. The last section is a short summary and discussion.

B. Two-fermionic-parameter bosonization

Firstly, we expand the component fields ξ and u in the form of

$$\xi(x, t) = p\zeta_1 + q\zeta_2, \quad (5a)$$

$$u(x, t) = u_0 + u_1 \zeta_1 \zeta_2, \quad (5b)$$

where ζ_1 and ζ_2 are two Grassmann parameters, while the coefficients $p \equiv p(x, t)$, $q \equiv q(x, t)$, $u_0 \equiv u_0(x, t)$ and $u_1 \equiv u_1(x, t)$ are four usual real or complex functions with respect to the spacetime variables x and t , then the sKdV system (4a)–(4b) is changed to

$$u_{0t} + u_{0xxx} + 6u_0 u_{0x} = 0, \quad (6a)$$

$$p_t + p_{xxx} + 3u_0 p_x + 3u_{0x} p = 0, \quad (6b)$$

$$q_t + q_{xxx} + 3u_0 q_x + 3u_{0x} q = 0, \quad (6c)$$

$$u_{1t} + u_{1xxx} + 6u_0 u_{1x} + 6u_{0x} u_1 = 3(pq_{xx} - qp_{xx}) \quad (6d)$$

that is just the bosonic-looking of the sKdV system (4) in two fermionic parameter case. Eq. (6a) is exactly the usual KdV equation which has been widely studied. Eqs. (6b) and (6c) are linear homogeneous in p and q respectively, and Eq. (6d) is linear nonhomogeneous in u_1 . Thereby, in principle, these equations can be easily solved. This is just one of the advantages of the bosonization approach.

Now let us consider the traveling wave solutions of the bosonic-looking equations (6). Introducing the traveling wave variable $X = kx + \omega t + c_0$ with constants k , ω and c_0 , the system (6a)–(6d) are transformed to the ordinary differential equations (ODEs)

$$k^3 u_{0XXX} + (6ku_0 + \omega)u_{0X} = 0, \quad (7a)$$

$$k^3 p_{XXX} + (3ku_0 + \omega)p_X + 3ku_{0X}p = 0, \quad (7b)$$

$$k^3 q_{XXX} + (3ku_0 + \omega)q_X + 3ku_{0X}q = 0, \quad (7c)$$

$$k^3 u_{1XXX} + (6ku_0 + \omega)u_{1X} + 6ku_{0X}u_1 = 3k^2(pq_{XX} - qp_{XX}). \quad (7d)$$

Remark. The traveling waves in the superspace, $\Phi(x, t, \theta) = \Phi(kx + \omega t + c_0 + \zeta\theta)$, with Grassmann constant ζ are different from those of in the usual space-time $\{x, t\}$. Hereafter, the traveling waves we discuss are only in the usual space-time $\{x, t\}$ but not in the superspace $\{x, t, \theta\}$.

Obviously, Eq. (7a) is the traveling wave reduction of the KdV equation, and its periodic wave solutions including solitary wave solutions are well known. To solve the ODE system (7a)–(7d), we try to build the mapping and deformation relation between the traveling wave solutions of the classical KdV equation and the sKdV equation, and then to construct the exact solutions of the sKdV equation by using the known solutions of the KdV equation.

We first solve out u_{0X} from Eq. (7a). The result reads

$$u_{0X} = \frac{a_0}{k^2} \sqrt{-k(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3)}, \quad (8)$$

where c_1 and c_2 are two integral constants and $a_0 = \pm 1$. The only linear inhomogeneous ODE (7d) can be directly integrated once, and becomes

$$k^3 u_{1XX} + (6ku_0 + \omega)u_1 = f(X), \quad (9)$$

where the inhomogeneous term is

$$f(X) = 3k^2(pq_X - qp_X) - b_0 \quad (10)$$

with an integral constant b_0 .

To get the mapping relations of p , q and u_1 , we introduce the variable transformations as follows

$$p(X) = P(u_0(X)), \quad q(X) = Q(u_0(X)), \quad u_1(X) = U_1(u_0(X)). \quad (11)$$

Using the transformation (11) and eliminating u_{0X} via Eq. (8), the linear ODEs (7b)–(7c) as well as (9) are changed to

$$(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3) \frac{d^3 P}{du_0^3} + 3(3ku_0^2 + \omega u_0 + c_1) \frac{d^2 P}{du_0^2} + 3ku_0 \frac{dP}{du_0} - 3kP = 0, \quad (12a)$$

$$(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3) \frac{d^3 Q}{du_0^3} + 3(3ku_0^2 + \omega u_0 + c_1) \frac{d^2 Q}{du_0^2} + 3ku_0 \frac{dQ}{du_0} - 3kQ = 0, \quad (12b)$$

$$(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3) \frac{d^2 U_1}{du_0^2} + (3ku_0^2 + \omega u_0 + c_1) \frac{dU_1}{du_0} - (6ku_0 + \omega)U_1 = F(u_0), \quad (12c)$$

where

$$F(u_0) = 3a_0 \left(Q \frac{dP}{du_0} - P \frac{dQ}{du_0} \right) \sqrt{-k(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3)} + b_0. \quad (13)$$

On this basis, the mapping and deformation relations are constructed as

$$P(u_0) = A_1 u_0 + A_2 \sqrt{ku_0^2 + c_1} \sin [R(u_0) + A_3], \quad (14a)$$

$$Q(u_0) = A_4 u_0 + A_5 \sqrt{ku_0^2 + c_1} \sin [R(u_0) + A_6], \quad (14b)$$

where A_1, A_2, A_3, A_4, A_5 and A_6 are arbitrary constants, and

$$R(u_0) = \int \frac{\sqrt{-c_1(c_2k^4 + c_1\omega)}}{(ku_0^2 + c_1) \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3}} du_0.$$

Following the relation (14), the solution for U_1 can be obtained from Eq. (12c) as

$$U_1(u_0) = \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3} \left[A_8 + \int \frac{A_7 + \int F(u_0)du_0}{(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3)^{3/2}} du_0 \right], \quad (15)$$

where A_7 and A_8 are two integral constants. Thus, we have obtained the general two-fermionic-parameter traveling wave solutions of the sKdV system

$$\begin{aligned} u &= u_0 + \zeta_1 \zeta_2 \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3} \left[A_8 + \int \frac{A_7 + \int F(u_0)du_0}{(2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3)^{3/2}} du_0 \right], \quad (16a) \\ \xi &= \zeta_1 \left\{ A_1 u_0 + A_2 \sqrt{ku_0^2 + c_1 \sin[R(u_0) + A_3]} \right\} + \zeta_2 \left\{ A_4 u_0 + A_5 \sqrt{ku_0^2 + c_1 \sin[R(u_0) + A_6]} \right\}, \quad (16b) \end{aligned}$$

with the known solution u_0 of the usual KdV equation.

For a special case, $A_2 = A_5 = A_7 = b_0 = 0$, the above traveling wave solution becomes

$$P = a_1 u_0, \quad (a_1 = A_1) \quad (17a)$$

$$Q = a_2 u_0, \quad (a_2 = A_4) \quad (17b)$$

and

$$U_1 = A_8 \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1u_0 - c_2k^3} = a_3 u_{0X}, \quad \left(a_3 = \frac{A_8}{a_0} \sqrt{-k^3} \right), \quad (18)$$

where the second equal sign of the above equation is due to the relation (8). It is interesting that the expression U_1 (18) is an ordinary type of the symmetries of the traveling wave equation (7a).

In fact, for any given $u_0(x, t)$ being a solution of the usual KdV equation, a certain type of solutions of the bosonic-looking equation (6) can be constructed as follows

$$p = a_1 u_0, \quad (19a)$$

$$q = a_2 u_0, \quad (19b)$$

$$u_1 = \sigma(u_0), \quad (19c)$$

where $\sigma(u_0)$ represents any symmetry of the usual KdV equation (6a).

Under the circumstances of describing p and q as the form of (19a)–(19b), u_0 can be chosen as any solution of the KdV equation. Then the first three equations of the bosonic-looking equations (6) are satisfied automatically. Obviously, the righthand side of the nonhomogeneous equation (6d) equals zero because of Eqs. (19a) and (19b). In such situations, u_1 from Eq. (6d) exactly satisfies

the symmetry equation of the usual KdV system (6a). This means that we have much freedom to choose u_0 so as to construct solutions of the sKdV equations. Furthermore, it is not limited to the traveling wave solutions of u_0 . It is worth mentioning that the KdV equation possesses infinitely many symmetries, and thus infinitely many u_1 can be generated. All in all, we can construct not only traveling wave solutions but also many other new types of solutions of the sKdV system by using the solutions and infinitely many symmetries of the KdV equation.

It is known that the solution (8) can be expressed by means of the Jacobi elliptic functions, say,

$$u_0 = -\frac{k^3(2m^2 - 1) + \omega}{6k} + \frac{k^2 m^2}{2} \text{cn}^2\left(\frac{kx + \omega t + c_0}{2}, m\right), \quad (20)$$

where the constants c_1 and c_2 are related to the other constants through

$$c_1 = \frac{\omega^2}{12k} - \frac{k^5}{12}(1 - m^2 + m^4),$$

and

$$c_2 = -\frac{\omega^3}{108k^5} + \frac{k\omega}{36}(1 - m^2 + m^4) - \frac{k^4}{108}(2 - 3m^2 - 3m^4 + 2m^6).$$

Therefore, we obtain a special type of exact solutions of the sKdV system

$$\begin{aligned} u = & -\frac{k^3(2m^2 - 1) + \omega}{6k} + \frac{k^2 m^2}{2} \text{cn}^2\left(\frac{kx + \omega t + c_0}{2}, m\right) \\ & + \zeta_1 \zeta_2 \left\{ C_1 J + C_2 [1 + 3k^3 m^2 t J] + C_3 [2(\omega - k^3(m^2 + 1)) + 6k^3 m^2 S^2 \right. \\ & + 3k^3 m^2 (3\omega t + kx) J] + C_4 [18k^6 m^4 S^4 - 12k^3 m^2 (2\omega + k^3(1 + m^2)) S^2 \\ & - 2(m^2 - 2)(2m^2 - 1)k^6 + 8\omega k^3(m^2 + 1) - 4\omega^2 + k^2(3k^2 m^2(k^3(m^2 + 1) - 4\omega)x \\ & - 3km^2(2k^6(1 - m^2 + m^4) + \omega k^3(2m^2 - 1) + 8\omega^2)t - 9k^5 m^4 \int S^2 dx) J] \\ & \left. + C_5 [6k^2 m^3 S^4 - 6mk^2 S^2 + (1 - m^2)mk^2 + 3mk^2((E + m^2 - 1)(kx + \omega t) + 2Z)J] \right\}, \quad (21a) \end{aligned}$$

$$\xi = (a_1 \zeta_1 + a_2 \zeta_2) \left[-\frac{k^3(2m^2 - 1) + \omega}{6k} + \frac{k^2 m^2}{2} \text{cn}^2\left(\frac{kx + \omega t + c_0}{2}, m\right) \right], \quad (21b)$$

where C_i ($i = 1, \dots, 5$), k , ω , a_1 , a_2 , c_0 , m are usual arbitrary constants, ζ_1 and ζ_2 are arbitrary Grassmann odd constants, $Z \equiv Z\left(\frac{1}{2}(kx + \omega t + c_0), m\right)$ is the Jacobi zeta function, $E = \frac{E(m)}{K(m)}$ is the ratio of the complete elliptic integral of the second kind to the first kind, and

$$S \equiv \text{sn}\left(\frac{kx + \omega t + c_0}{2}, m\right), \quad J \equiv \text{cn}\left(\frac{kx + \omega t + c_0}{2}, m\right) \text{dn}\left(\frac{kx + \omega t + c_0}{2}, m\right) S.$$

It is worth to mention that the solution (21a) is neither a traveling nor a periodic wave solution for $C_2 C_3 C_4 \neq 0$.

It is noted that for the solution (21), the soliton limit, $m = 1$, $\omega = -k^3$, exists for $C_2 = C_5 = 0$. The result reads ($\eta \equiv \frac{k}{2}(x - k^2t - x_0)$, $T \equiv \tanh(\eta)$)

$$u = \zeta_1 \zeta_2 \left\{ \left[k(n_2 + 2n_3)x - k^3(3n_2 + 4n_3)t + n_1 + n_3 \ln \frac{1-T}{1+T} \right] T - 2n_2 - 2n_3 \right\} \text{sech}^2(\eta) + \frac{k^2}{2} \text{sech}^2(\eta), \quad (22a)$$

$$\xi = (a_1 \zeta_1 + a_2 \zeta_2) \text{sech}^2(\eta), \quad (22b)$$

with the usually arbitrary constants $\{k, a_1, a_2, x_0, n_1, n_2, n_3\}$, and arbitrary Grassmann odd constants $\{\zeta_1, \zeta_2\}$.

The solution (5) with (19) extends every solution of the KdV equation to many special types of solutions for the sKdV system, due to the existence of the infinitely many symmetries of the KdV equation. For instance, the N-soliton solution of the KdV equation reads

$$u_{KdV} = 2 \left\{ \ln \left[1 + \sum_{k=1}^N \sum_{i_1 > i_2 > \dots > i_k} \prod_{m > n} A_{i_m i_n} \exp \left(\sum_{j=1}^k \eta_{i_j} \right) \right] \right\}_{xx} \quad (23)$$

with arbitrary constants $\{k_i, \eta_{0i}, i = 1, 2, \dots, N\}$ and

$$A_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad \eta_j = k_j x - k_j^3 t + \eta_{0j}.$$

Correspondingly, a special type of multiple soliton solutions of the sKdV (4) can be simply written as

$$u = u_{KdV} + \zeta_1 \zeta_2 \sum_{i=0}^N \left(B_i \frac{\partial}{\partial \eta_{0i}} u_{KdV} + M_i \frac{\partial}{\partial k_i} u_{KdV} \right), \quad (24a)$$

$$\xi = (a_1 \zeta_1 + a_2 \zeta_2) u_{KdV} \quad (24b)$$

with further arbitrary constants B_i and M_i for $i = 1, 2, \dots, N$.

C. Three-fermionic-parameter bosonization

In the case of three Grassmann parameters ζ_1, ζ_2 and ζ_3 , the component fields ξ and u are expanded as

$$\xi(x, t) = p_1 \zeta_1 + p_2 \zeta_2 + p_3 \zeta_3 + p_4 \zeta_1 \zeta_2 \zeta_3, \quad (25a)$$

$$u(x, t) = u_0 + u_1 \zeta_2 \zeta_3 + u_2 \zeta_3 \zeta_1 + u_3 \zeta_1 \zeta_2, \quad (25b)$$

where the coefficients $p_i \equiv p_i(x, t)$ ($i = 1, 2, 3, 4$) and $u_j \equiv u_j(x, t)$ ($j = 0, 1, 2, 3$) are eight real or complex bosonic functions of the indicated variables. Then the sKdV system (4a)–(4b) is changed to

$$u_{0t} + u_{0xxx} + 6u_0u_{0x} = 0, \quad (26a)$$

$$p_{1t} + p_{1xxx} + 3u_0p_{1x} + 3u_{0x}p_1 = 0, \quad (26b)$$

$$p_{2t} + p_{2xxx} + 3u_0p_{2x} + 3u_{0x}p_2 = 0, \quad (26c)$$

$$p_{3t} + p_{3xxx} + 3u_0p_{3x} + 3u_{0x}p_3 = 0, \quad (26d)$$

$$u_{1t} + u_{1xxx} + 6u_0u_{1x} + 6u_{0x}u_1 = 3(p_2p_{3xx} - p_3p_{2xx}), \quad (26e)$$

$$u_{2t} + u_{2xxx} + 6u_0u_{2x} + 6u_{0x}u_2 = 3(p_3p_{1xx} - p_1p_{3xx}), \quad (26f)$$

$$u_{3t} + u_{3xxx} + 6u_0u_{3x} + 6u_{0x}u_3 = 3(p_1p_{2xx} - p_2p_{1xx}), \quad (26g)$$

$$p_{4t} + p_{4xxx} + 3u_0p_{4x} + 3u_{0x}p_4 = -3(u_1p_1 + u_2p_2 + u_3p_3)_x. \quad (26h)$$

Just similar to the previous case, the system (26a)–(26h) also has no fermionic quantities. Besides, Eq. (26a) is exactly the KdV equation. The rest seven equations are linear in p_i ($i = 1, 2, 3, 4$), and u_l ($l = 1, 2, 3$), respectively. It is observed that the number of the inhomogeneous equations increases, so that this bosonic-looking of the sKdV system is somewhat complex.

Introducing the traveling wave variable $X = kx + \omega t + c_0$, where k , ω and c_0 are arbitrary constants, the bosonization system (26) becomes

$$k^3u_{0XXX} + (6ku_0 + \omega)u_{0X} = 0, \quad (27a)$$

$$k^3p_{1XXX} + (3ku_0 + \omega)p_{1X} + 3ku_{0X}p_1 = 0, \quad (27b)$$

$$k^3p_{2XXX} + (3ku_0 + \omega)p_{2X} + 3ku_{0X}p_2 = 0, \quad (27c)$$

$$k^3p_{3XXX} + (3ku_0 + \omega)p_{3X} + 3ku_{0X}p_3 = 0, \quad (27d)$$

$$k^3u_{1XXX} + (6ku_0 + \omega)u_{1X} + 6ku_{0X}u_1 = 3k^2(p_2p_{3XX} - p_3p_{2XX}), \quad (27e)$$

$$k^3u_{2XXX} + (6ku_0 + \omega)u_{2X} + 6ku_{0X}u_2 = 3k^2(p_3p_{1XX} - p_1p_{3XX}), \quad (27f)$$

$$k^3u_{3XXX} + (6ku_0 + \omega)u_{3X} + 6ku_{0X}u_3 = 3k^2(p_1p_{2XX} - p_2p_{1XX}), \quad (27g)$$

$$k^3p_{4XXX} + (3ku_0 + \omega)p_{4X} + 3ku_{0X}p_4 = -3k(u_1p_1 + u_2p_2 + u_3p_3)_X. \quad (27h)$$

It is quite obvious that Eq. (27a) is the same as (7a), while Eqs. (27b)–(27d) have an analogy with (7b)–(7c) and (27e)–(27g) with (7d). Coefficients of the left-hand side of the last equation (27h)

is consistent with Eqs. (27b)–(27d), but its right-hand side is related to p_l and u_l ($l = 1, 2, 3$), not always zero.

To solve the ODE system (27a)–(27h), following the approach adopted in the previous section, we first solve p_l and u_l . Integrating the inhomogeneous ODEs (27e)–(27h) once, we have

$$k^3 u_{lXX} + (6ku_0 + \omega)u_l = f_l(X), \quad (28a)$$

$$k^3 p_{4XX} + (3ku_0 + \omega)p_4 = f_4(X), \quad (28b)$$

where

$$\begin{aligned} f_1(X) &= 3k^2(p_2 p_{3X} - p_3 p_{2X}) - b_1, \\ f_2(X) &= 3k^2(p_3 p_{1X} - p_1 p_{3X}) - b_2, \\ f_3(X) &= 3k^2(p_1 p_{2X} - p_2 p_{1X}) - b_3, \\ f_4(X) &= -3k(u_1 p_1 + u_2 p_2 + u_3 p_3) - b_4, \end{aligned} \quad (29)$$

with constants b_1, b_2, b_3 and b_4 .

Considering the variable transformations

$$\begin{aligned} u_l(X) &= U_l(u_0(X)), \quad (l = 1, 2, 3), \\ p_i(X) &= P_i(u_0(X)), \quad (i = 1, 2, 3, 4) \end{aligned} \quad (30)$$

and using (8) to eliminate u_{0X} , we can transform the linear ODEs (27b)–(27d) and (28) to

$$(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3) \frac{d^3 P_l}{du_0^3} + 3(3ku_0^2 + \omega u_0 + c_1) \frac{d^2 P_l}{du_0^2} + 3ku_0 \frac{dP_l}{du_0} - 3kP_l = 0, \quad (31a)$$

$$(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3) \frac{d^2 U_l}{du_0^2} + (3ku_0^2 + \omega u_0 + c_1) \frac{dU_l}{du_0} - (6ku_0 + \omega)U_l = F_l(u_0), \quad (31b)$$

$$(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3) \frac{d^2 P_4}{du_0^2} + (3ku_0^2 + \omega u_0 + c_1) \frac{dP_4}{du_0} - (3ku_0 + \omega)P_4 = F_4(u_0), \quad (31c)$$

where

$$\begin{aligned} F_1(u_0) &= 3a_0 \left(P_3 \frac{dP_2}{du_0} - P_2 \frac{dP_3}{du_0} \right) \sqrt{-k(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3)} + b_1 \\ F_2(u_0) &= 3a_0 \left(P_1 \frac{dP_3}{du_0} - P_3 \frac{dP_1}{du_0} \right) \sqrt{-k(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3)} + b_2, \\ F_3(u_0) &= 3a_0 \left(P_2 \frac{dP_1}{du_0} - P_1 \frac{dP_2}{du_0} \right) \sqrt{-k(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3)} + b_3, \\ F_4(u_0) &= 3k(U_1 P_1 + U_2 P_2 + U_3 P_3) + b_4. \end{aligned} \quad (32)$$

By repeating the processes in the last section, the general three-fermionic-parameter traveling wave solutions for the sKdV system can be derived

$$\begin{aligned}
u &= u_0 + u_{0X} \sum_{l=1}^3 \zeta_l \zeta_{l+1} \left[h_l + \int \frac{g_l + \int F_l(u_0) du_0}{(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3)^{3/2}} du_0 \right], \\
\xi &= \sum_{l=1}^3 \zeta_l \left\{ r_l u_0 + s_l \sqrt{ku_0^2 + c_1} \sin [R(u_0) + \alpha_l] \right\} \\
&\quad + \zeta_1 \zeta_2 \zeta_3 \left\{ r_4 u_0 + s_4 \sqrt{ku_0^2 + c_1} \sin [R(u_0) + \alpha_4] \right. \\
&\quad \left. + \sqrt{ku_0^2 + c_1} \int^{u_0} \frac{\sin [R(u_0) - R(y)] F_4(y) \sqrt{ky^2 + c_1}}{\sqrt{c_1(c_2 k^4 + c_1 \omega)(c_2 k^3 - 2ky^3 - \omega y^2 - 2c_1 y)}} dy \right\}, \tag{33}
\end{aligned}$$

where $\{g_l, h_l, r_l, r_4, s_l, s_4, \alpha_l, \alpha_4\}$ are arbitrary constants and $\zeta_4 = \zeta_1$.

Similar to the two fermionic parameter case, for nontraveling wave solutions of Eq. (26), we just write down a special case with

$$p_l = d_l u_0, \quad (l = 1, 2, 3, 4), \tag{34a}$$

$$u_1 = \sigma_1(u_0), \tag{34b}$$

$$u_2 = \sigma_2(u_0), \tag{34c}$$

$$u_3 = -d_3^{-1}(d_1 u_1 + d_2 u_2), \tag{34d}$$

where d_1, d_2, d_3, d_4 are constants, u_0 is an arbitrary solution of the usual KdV equation, $\sigma_1(u_0)$ and $\sigma_2(u_0)$ are arbitrary symmetries of the usual KdV equation. Finally, the sKdV system (4) possesses the following special solution

$$u = u_0 + \sigma_1(u_0) \zeta_2 \zeta_3 + \sigma_2(u_0) \zeta_3 \zeta_1 - d_3^{-1} [d_1 \sigma_1(u_0) + d_2 \sigma_2(u_0)] \zeta_1 \zeta_2, \tag{35a}$$

$$\xi = (d_1 \zeta_1 + d_2 \zeta_2 + d_3 \zeta_3 + d_4 \zeta_1 \zeta_2 \zeta_3) u_0 \tag{35b}$$

with an arbitrary solution u_0 , two arbitrary symmetries $\sigma_1(u_0)$ and $\sigma_2(u_0)$ of the usual KdV equation, three Grassmann numbers ζ_i ($i = 1, 2, 3$) and four arbitrary usual real constants d_l ($l = 1, 2, 3, 4$). When one of the ζ_i tends to zero, the solution (35) turns back to that of the last section for two fermionic parameters.

Actually, applying the similar procedure for any numbers of the fermionic parameters, one can obtain various exact solutions such as the general traveling wave solution and the special solutions like Eq. (35).

D. N-fermionic-parameter bosonization

For the supersymmetric system introduced $N \geq 2$ fermionic parameters ζ_i ($i = 1, 2, \dots, N$), the component fields u and ξ can be expanded as

$$u(x, t) = u_0 + \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n} \leq N} u_{i_1 i_2 \dots i_{2n}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n}}, \quad (36a)$$

$$\xi(x, t) = \sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n-1} \leq N} v_{i_1 i_2 \dots i_{2n-1}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n-1}}, \quad (36b)$$

where the coefficients $u_0 \equiv u_0(x, t)$, $u_{i_1 i_2 \dots i_{2n}} \equiv u_{i_1 i_2 \dots i_{2n}}(x, t)$ ($1 \leq i_1 < i_2 < \dots < i_{2n} \leq N$) and $v_{i_1 i_2 \dots i_{2n-1}} \equiv v_{i_1 i_2 \dots i_{2n-1}}(x, t)$ ($1 \leq i_1 < i_2 < \dots < i_{2n-1} \leq N$) are 2^N real or complex bosonic functions of classical spacetime variable x and t . Substituting Eq. (36) into the sKdV model (4), we obtain the following bosonic system of 2^N equations

$$u_{0t} + u_{0xxx} + 6u_0 u_{0x} = 0, \quad (37a)$$

$$L_o v_{i_1 i_2 \dots i_{2n-1}} = \begin{cases} 0 & \text{for } n = 1 \\ -3 \sum_{W_1} (-1)^{\tau(j_1, j_2, \dots, j_{2n-1})} [u_{i_{j_1} i_{j_2} \dots i_{j_{2l}} v_{i_{j_{2l+1}} i_{j_{2l+2}} \dots i_{j_{2n-1}}}]_x & \text{for } n = 2, 3, \dots, \lfloor \frac{N+1}{2} \rfloor \end{cases}, \quad (37b)$$

$$L_e u_{i_1 i_2 \dots i_{2n}} = \begin{cases} 3 \sum_{W_2} (-1)^{\tau(j_1, j_2)} [v_{i_{j_1}} (v_{i_{j_2}})_x]_x, & \text{for } n = 1 \\ 3 \sum_{W_2} (-1)^{\tau(j_1, j_2, \dots, j_{2n})} [v_{i_{j_1} i_{j_2} \dots i_{j_{2l-1}}} (v_{i_{j_{2l}} i_{j_{2l+1}} \dots i_{j_{2n}})_x]_x \\ -3 \sum_{W_3} (-1)^{\tau(j_1, j_2, \dots, j_{2n})} [u_{i_{j_1} i_{j_2} \dots i_{j_{2l}} u_{i_{j_{2l+1}} i_{j_{2l+2}} \dots i_{j_{2n}}}]_x & \text{for } n = 2, 3, \dots, \lfloor \frac{N}{2} \rfloor, \end{cases}, \quad (37c)$$

where

$$\tau(j_1, j_2, \dots, j_N) = \begin{cases} 0 & j_1, j_2, \dots, j_N \text{ is even permutation} \\ 1 & j_1, j_2, \dots, j_N \text{ is odd permutation} \end{cases},$$

$$W_1 = \{(j_1, j_2, \dots, j_{2n-1}) | 1 \leq j_1 < j_2 < \dots < j_{2l} \leq 2n-1, 1 \leq j_{2l+1} < j_{2l+2} < \dots < j_{2n-1} \leq 2n-1, \\ 1 \leq l \leq n-1, j_{h_1} \neq j_{h_2} (h_1 \neq h_2)\},$$

$$W_2 = \{(j_1, j_2, \dots, j_{2n}) | 1 \leq j_1 < j_2 < \dots < j_{2l-1} \leq 2n, 1 \leq j_{2l} < j_{2l+1} < \dots < j_{2n} \leq 2n, \\ 1 \leq l \leq n, j_{h_1} \neq j_{h_2} (h_1 \neq h_2)\},$$

$$W_3 = \{(j_1, j_2, \dots, j_{2n}) | 1 \leq j_1 < j_2 < \dots < j_{2l} \leq 2n, 1 \leq j_{2l+1} < j_{2l+2} < \dots < j_{2n} \leq 2n, \\ 1 \leq l \leq n-1, j_{h_1} \neq j_{h_2} (h_1 \neq h_2)\},$$

and two operators read

$$L_e(u_0) = \partial_t + \partial_{xxx} + 6u_0\partial_x + 6u_{0x},$$

$$L_o(u_0) = \partial_t + \partial_{xxx} + 3u_0\partial_x + 3u_{0x}.$$

Similarly to the section, the general traveling wave solution of the sKdV equation (3) with N fermionic parameters can be written as

$$\Phi = \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n-1} \leq N} v_{i_1 i_2 \dots i_{2n-1}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n-1}} + \theta(u_0 + \sum_{n=1}^{\lfloor \frac{N+1}{2} \rfloor} \sum_{1 \leq i_1 < \dots < i_{2n} \leq N} u_{i_1 i_2 \dots i_{2n}} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_{2n}}), \quad (38)$$

where

$$\begin{aligned} v_{i_1 i_2 \dots i_{2n-1}} &= V_{i_1 i_2 \dots i_{2n-1}}(u_0) \\ &= r_{i_1 i_2 \dots i_{2n-1}} u_0 + s_{i_1 i_2 \dots i_{2n-1}} \sqrt{ku_0^2 + c_1} \sin[R(u_0) + \alpha_{i_1 i_2 \dots i_{2n-1}}] \\ &\quad + \sqrt{ku_0^2 + c_1} \int^{u_0} \frac{\sin[R(u_0) - R(y)] E_{i_1 i_2 \dots i_{2n-1}}(y) \sqrt{ky^2 + c_1}}{\sqrt{c_1(c_2 k^4 + c_1 \omega)(c_2 k^3 - 2ky^3 - \omega y^2 - 2c_1 y)}} dy, \end{aligned} \quad (39a)$$

$$\begin{aligned} u_{i_1 i_2 \dots i_{2n}} &= U_{i_1 i_2 \dots i_{2n}}(u_0) \\ &= \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3} \left[h_{i_1 i_2 \dots i_{2n}} + \int \frac{g_{i_1 i_2 \dots i_{2n}} + \int F_{i_1 i_2 \dots i_{2n}}(u_0) du_0}{(2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3)^{3/2}} du_0 \right], \end{aligned} \quad (39b)$$

with

$$\begin{aligned} E_{i_1 i_2 \dots i_{2n-1}}(u_0) &= \begin{cases} 0 & \text{if } n = 1 \\ 3k \sum_{W_1} (-1)^{\tau(j_1, j_2, \dots, j_{2n-1})} U_{i_{j_1} i_{j_2} \dots i_{j_{2l}}} V_{i_{j_{2l+1}} i_{j_{2l+2}} \dots i_{j_{2n-1}}} + b_{i_1 i_2 \dots i_{2n-1}} & \text{if } n = 2, 3, \dots, \frac{N+1}{2} \end{cases}, \\ F_{i_1 i_2 \dots i_{2n}}(u_0) &= \begin{cases} -3k^2 u_{0x} \sum_{W_2} (-1)^{\tau(j_1, j_2)} V_{i_{j_1}} (V_{i_{j_2}})_{u_0} + b_{i_1 i_2} & \text{if } n = 1 \\ 3k \sum_{W_3} (-1)^{\tau(j_1, j_2, \dots, j_{2n})} U_{i_{j_1} i_{j_2} \dots i_{j_{2l}}} U_{i_{j_{2l+1}} i_{j_{2l+2}} \dots i_{j_{2n}}} & \\ -3k^2 u_{0x} \sum_{W_2} (-1)^{\tau(j_1, j_2, \dots, j_{2n})} V_{i_{j_1} i_{j_2} \dots i_{j_{2l-1}}} (V_{i_{j_{2l}} i_{j_{2l+1}} \dots i_{j_{2n}}})_{u_0} + b_{i_1 i_2 \dots i_{2n}} & \text{if } n = 2, 3, \dots, \frac{N}{2} \end{cases}, \end{aligned} \quad (40)$$

and

$$R(u_0) = \int \frac{\sqrt{-c_1(c_2 k^4 + c_1 \omega)}}{(ku_0^2 + c_1) \sqrt{2ku_0^3 + \omega u_0^2 + 2c_1 u_0 - c_2 k^3}} du_0,$$

where u_0 represents the solution of KdV equation (37a), $r_{i_1 i_2 \dots i_{2n-1}}$, $s_{i_1 i_2 \dots i_{2n-1}}$, $\alpha_{i_1 i_2 \dots i_{2n-1}}$, $h_{i_1 i_2 \dots i_{2n}}$, $g_{i_1 i_2 \dots i_{2n}}$, $b_{i_1 i_2 \dots i_{2n-1}}$, $b_{i_1 i_2 \dots i_{2n}}$ are arbitrary constants.

E. Conclusions

In summary, a simple bosonization approach to deal with supersymmetric system is developed. With n fermionic parameters, the bosonization procedure of the supersymmetric systems has been successfully applied to the sKdV equation in detail. Such an integrable nonlinear system is simplified to the usual KdV equation together with several linear differential equations without fermionic variables. The traveling wave solutions of the bosonization systems can be obtained simply by integrations, say, (16) and (33) for the two and three fermionic parameter cases, respectively.

Some special types of exact supersymmetric extensions of any solutions of the usual KdV equation can be obtained straightforwardly through the exact solutions of the KdV equation and the related symmetries (for instance, (21) and (35)). The general extensions of the KdV solutions to those of supersymmetric form are obtained by solving some linear systems with variable coefficients depending on the usual exact KdV solutions.

From the procedure exhibited in this paper, we can conclude that the bosonization approach can be applicable to not only the supersymmetric integrable systems but also all the models with fermion fields no matter they are integrable or not.

It should be emphasized that the solutions obtained via the bosonization procedure are completely different from those obtained via other methods such as the bilinear approach [21]. Especially, the traditional multisoliton solutions of the sKdV are different from ours, say, Eq. (24). This fact shows us that for the sKdV equation there exist various kinds of localized excitations. In other words, in addition to the single supersymmetric traveling wave soliton solution (in the super space-time $\{x, t, \theta\}$) known in literature [21], there are infinitely many single traveling soliton extensions in the usual space-time $\{x, t\}$.

The abundant property of the soliton excitations of the classical sKdV reveals some open problems in the both classical and quantum theories. In the classical level, one of the most important problems may be how to further develop the bosonization procedure such that the known solutions obtained in other approaches can also be included in. In other words, How to developed a bosonization procedure such that the bosonized system is completely equivalent to the classic supersymmetric one. In the quantum level, three of the important topics should be mentioned to investigated: How to reflect the richness of the localized excitations in the usual quantization procedure of the supersymmetry models[2]? Can we find a quantized bosonization approach for

supersymmetric integrable systems? What is about the quantization versions of the bosonized systems of this paper?

F. Acknowledgement

The work was sponsored by the National Natural Science Foundation of China (Nos. 10735030 and 10905038), the National Basic Research Programs of China (973 Programs 2007CB814800 and 2005CB422301) and K. C. Wong Magna Fund in Ningbo University.

-
- [1] D. J. Gross and A. Migdal, Nucl. Phys. B 34 (1990) 333; M. Douglass, Phys. Lett. B 238 (1990) 176; R. Dijkgraaf and E. Witten, Nucl. Phys. B 342 (1990) 486.
 - [2] P. P. Kulish, A. M. Zeitlin, Phys. Lett. B.597 (2004) 229; Nucl. Phys. B 709 (2005) 578; Nucl. Phys. B 720 (2005) 289.
 - [3] K. B. Efetov, C. Pepin and H. Meier, Phys. Rev. Lett. 103 (2009) 186403; A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998); T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, Oxford, 2003); A. Luther, Phys. Rev. B 19 (1979) 320.
 - [4] O. Oguz and D. Yazici, Int. J. Mod. Phys. A 25 (2010) 1069; B. Kupershmidt, Phys. Lett. A 102 (1984) 213-215.
 - [5] K. Tian, Q. Liu, Phys. Lett. A 373 (2009) 1807; A. J. Hariton, J. Phys. A: Math. Gen. 39 (2006) 7105-7114; P. Labelle and P. Mathieu, J. Math. Phys. 32 (1991) 923; C. Laberge, P. Mathieu, Phys. Lett. B 215 (1988) 718.
 - [6] Mousumi Saha, A. Roy Chowdhury, International Journal of Theoretical Physics, 38 (1999) 2037-2047.
 - [7] S. Andrea, A. Restuccia, A. Sotomayor, J. Math. Phys. 42 (2001) 2625-2634.
 - [8] Yu. I. Maniu, A. O. Radul, Commun. Math. Phys. 98 (1985) 65-67.
 - [9] P. Mathieu, J. Math. Phys. 28 (1988) 2499-2506.
 - [10] P. Mathieu, Phys. Lett. B 203 (1988) 287-291.
 - [11] P. Mathieu, Phys. Lett. A 128 (1988) 169.
 - [12] W. Oevel, Z. Popowicz, Commun. Math. Phys. 139 (1991) 441-460.

- [13] J. M. Figueroa-Ofarrill, J. Mas, *Rev. Math. Phys.* 3 (1993) 479.
- [14] Q. P. Liu, *Lett. Math. Phys.* 35 (1995) 115.
- [15] A. S. Carstea, *Nonlinearity* 13 (2000) 1645-1656.
- [16] A. S. Carstea, A. Ramani, B. Grammaticos, *Nonlinearity* 14 (2001) 1419-1423.
- [17] Q. P. Liu, M. Manas, *Phys. Lett. B* 396 (1997) 133-140.
- [18] Q. P. Liu, Y. F. Xie, *Phys. Lett. A* 325 (2004) 139.
- [19] Q. P. Liu, X. B. Hu, *J. Phys. A: Math. Gen.* 38 (2005) 6371-6378.
- [20] S. Andrea, A. Restuccia, A. Sotomayor, *J. Math. Phys.* 46, 103517 (2005).
- [21] Y. C. Hon and E. G. Fan, *Theoret. Mat. Fiz.* 166 (2011) 366-387; E. G. Fan and Y. C. Hon, *Stud. Appl. Math.* 125 (2010) 343-373.